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Asymptotic completeness for N-body quantum systems with long-range interactions in a time-periodic electric field

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1 Introduction

In this article, we study the scattering theory for N-body quantum systems with long-range pair interactions in a time-periodic electric field whose mean in time is non-zero, where $N \geq 2$. We describe the results obtained in [A4] on the asymptotic completeness for such systems.

We consider a system of N particles moving in a given time-periodic electric field $\mathscr{E}(t) \in \boldsymbol{R}^d$, $\mathscr{E}(t) \not\equiv 0$. We suppose that $\mathscr{E}(t) \in C^0(\boldsymbol{R}; \boldsymbol{R}^d)$ has a period $T > 0$, that is, $\mathscr{E}(t+T) = \mathscr{E}(t)$ for any $t \in \boldsymbol{R}$, and its mean $\mathscr{E}$ in time is non-zero, i.e.

$$\mathscr{E} = \frac{1}{T}\int_0^T \mathscr{E}(t)\,dt \neq 0.$$

Let m_j, e_j and $r_j \in \boldsymbol{R}^d$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j-th particle, respectively. We suppose that the particles under consideration interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq N$. We assume that these pair potentials are independent of time t. Then the total Hamiltonian for the system is given by

$$\tilde{H}(t) = \sum_{1\leq j\leq N}\left\{-\frac{1}{2m_j}\Delta_{r_j} - e_j\langle \mathscr{E}(t), r_j\rangle\right\} + \sum_{1\leq j<k\leq N} V_{jk}(r_j - r_k),$$

where $\langle \xi, \eta\rangle = \sum_{j=1}^d \xi_j\eta_j$ for $\xi, \eta \in \boldsymbol{R}^d$. $\sum_{1\leq j<k\leq N} V_{jk}(r_j - r_k)$ will be written as V later. We now separate the part associated with the center of mass motion from $\tilde{H}(t)$ by standard procedure: We equip $\boldsymbol{R}^{d\times N}$ with the metric $r\cdot\tilde{r} = \sum_{j=1}^N m_j\langle r_j, \tilde{r}_j\rangle$ for $r = (r_1, \ldots, r_N)$, $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \boldsymbol{R}^{d\times N}$. We usually write $r \cdot r$ as r^2. We put $|r| = \sqrt{r^2}$. Let X be the configuration space in the center-of-mass frame:

$$X = \left\{ r \in \boldsymbol{R}^{d\times N} \;\middle|\; \sum_{1\leq j\leq N} m_j r_j = 0\right\}.$$

$\pi : \boldsymbol{R}^{d\times N} \to X$ denotes the orthogonal projection onto X. We put $x = \pi r$ for $r \in \boldsymbol{R}^{d\times N}$, and

$$E(t) = \pi\left(\frac{e_1}{m_1}\mathscr{E}(t), \ldots, \frac{e_N}{m_N}\mathscr{E}(t)\right), \quad E = \frac{1}{T}\int_0^T E(t)\,dt.$$

Throughout this article, we assume that there exists at least one pair (j, k) whose specific charges are different, that is, $e_j/m_j \neq e_k/m_k$. By virtue of this assumption, one sees that $E(t) \neq 0$ whenever $\mathscr{E}(t) \neq 0$, and that $E \neq 0$. By separating the part associated with the center of mass motion from $\tilde{H}(t)$, we obtain the Hamiltonian

$$H(t) = -\frac{1}{2}\Delta - E(t)\cdot x + V$$

on $L^2(X)$, where Δ is the Laplace-Beltrami operator on X. We will study the scattering theory for this Hamiltonian $H(t)$.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\cup_{1\le j\le m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. $\#(a)$ denotes the number of clusters in a. Let $\mathscr{A}$ be the set of all cluster decompositions. Suppose a, $b \in \mathscr{A}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any a is regarded as a refinement of itself. The one and N-cluster decompositions are denoted by $a_{\max}$ and $a_{\min}$, respectively. The pair (j, k) is identified with the $(N-1)$-cluster decomposition $\{(j, k), (1), \dots, \widehat{(j)}, \dots, \widehat{(k)}, \dots, (N)\}$.

Next we introduce two subspaces X^a and X_a of X for $a \in \mathscr{A}$:

$$X^a = \left\{ r \in X \;\middle|\; \sum_{j\in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a \right\}, \quad X_a = X \ominus X^a.$$

In particular, $X^{(j,k)}$ is identified with the configuration space for the relative position of j-th and k-th particles. Hence one can put $V_{(j,k)}(x^{(j,k)}) = V_{jk}(r_j - r_k)$. It is well known that $X_a = \{r \in X \mid r_j = r_k \text{ for each pair } (j,k) \subset a\}$, and that $L^2(X)$ is decomposed into $L^2(X^a) \otimes L^2(X_a)$. $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ denote the orthogonal projections onto X^a and X_a, respectively. We put $x^a = \pi^a x$ and $x_a = \pi_a x$ for $x \in X$. We now define the cluster Hamiltonian

$$H_a(t) = -\frac{1}{2}\Delta - E(t) \cdot x + V^a, \quad V^a = \sum_{(j,k)\subset a} V_{(j,k)}(x^{(j,k)}),$$

which governs the motion of the system broken into non-interacting clusters of particles. The intercluster potential I_a is given by

$$I_a(x) = V(x) - V^a(x) = \sum_{(j,k)\not\subset a} V_{(j,k)}(x^{(j,k)}).$$

Put $E^a(t) = \pi^a E(t)$ and $E_a(t) = \pi_a E(t)$. Then the cluster Hamiltonian $H_a(t)$ acting on $L^2(X)$ is decomposed into

$$H_a(t) = H^a(t) \otimes \mathrm{Id} + \mathrm{Id} \otimes T_a(t)$$

on $L^2(X^a) \otimes L^2(X_a)$, where Id are the identity operators,

$$H^a(t) = -\frac{1}{2}\Delta^a - E^a(t) \cdot x^a + V^a, \quad T_a(t) = -\frac{1}{2}\Delta_a - E_a(t) \cdot x_a,$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a).

Now we will state the assumptions on the pair potentials. Let c stand for a maximal element of the set $\{a \in \mathscr{A} \mid E^a = 0\}$ with respect to the relation $\subset$, where $E^a = \pi^a E$. Such a cluster decomposition uniquely exists, and it follows that $(j,k) \subset c$ is equivalent to $e_j/m_j = e_k/m_k$. If, in particular, $e_j/m_j \ne e_k/m_k$ for any $(j,k) \in \mathscr{A}$, then $c = a_{\min}$. Since $E \ne 0$ as mentioned above, we see that $c \ne a_{\max}$. We will impose different assumptions on V_{jk} according as $(j,k) \subset c$ or $(j,k) \not\subset c$: Let $\rho > 0$.

$(V)_{c,L}$ $V_{jk}(r) \in C^\infty(\boldsymbol{R}^d)$, $(j,k) \subset c$, is a real-valued function and satisfies

$$|\partial^\beta V_{jk}(r)| \leq C_\beta \langle r \rangle^{-(\rho'+|\beta|)}$$

with $\sqrt{3}-1 < \rho' \leq 1$.

$(V)_{\bar{c},G}$ $V_{jk}(r) \in C^\infty(\boldsymbol{R}^d)$, $(j,k) \not\subset c$, is a real-valued function and satisfies

$$|\partial^\beta V_{jk}(r)| \leq C_\beta \langle r \rangle^{-(\rho_G+|\beta|)}, \quad |\beta| \leq 1,$$
$$|\partial^\beta V_{jk}(r)| \leq C_\beta, \quad |\beta| \geq 2,$$

with $0 < \rho_G \leq 1/2$.

$(V)_{\bar{c},D,\rho}$ $V_{jk}(r) \in C^\infty(\boldsymbol{R}^d)$, $(j,k) \not\subset c$, is a real-valued function and satisfies

$$|\partial^\beta V_{jk}(r)| \leq C_\beta \langle r \rangle^{-(\rho+|\beta|/2)}.$$

Under these assumptions, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. Their closures are denoted by the same notations. If V_{jk}, $(j,k) \subset c$, satisfies $(V)_{c,L}$, then V_{jk} is called a long-range potential. We note that if V_{jk}, $(j,k) \not\subset c$, satisfies $(V)_{\bar{c},G}$ or $(V)_{\bar{c},D,\rho}$ with $\rho \leq 1/2$, then V_{jk} should be called a "Stark long-range" potential.

To formulate the obtained results precisely, we will define modified wave operators: Let $U(t,s)$, $U_a(t,s)$ and $\bar{U}_a(t,s)$, $a \subset c$, be unitary propagators generated by time-dependent Hamiltonians $H(t)$, $H_a(t)$ and $T_a(t)$, respectively. The existence and uniqueness of $U(t,s)$ are guaranteed by virtue of results of Yajima [Ya2] and the Avron-Herbst formula [CFKS] as follows: We introduce a strongly continuous family of unitary operators on $L^2(X)$ by

$$\tilde{\mathscr{T}}(t) = e^{-i\tilde{a}(t)} e^{i\tilde{b}(t)\cdot x} e^{-i\tilde{c}(t)\cdot p}, \tag{1.1}$$

where

$$\tilde{b}(t) = \int_0^t E(\tau)\, d\tau, \quad \tilde{c}(t) = \int_0^t \tilde{b}(\tau)\, d\tau, \quad \tilde{a}(t) = \frac{1}{2}\int_0^t \tilde{b}(\tau)^2\, d\tau. \tag{1.2}$$

We also introduce the time-dependent Hamiltonian $H^{Sc}(t)$ on $L^2(X)$ by

$$H^{Sc}(t) = -\frac{1}{2}\Delta + V(x + \tilde{c}(t)).$$

Since the propagator generated by $H^{Sc}(t)$ exists uniquely by virtue of results of [Ya2], we write it as $U^{Sc}(t,s)$. Then one sees that the propagator $U(t,s)$ generated by $H(t)$ also exists uniquely by virtue of the Avron-Herbst formula

$$U(t,s) = \tilde{\mathscr{T}}(t) U^{Sc}(t,s) \tilde{\mathscr{T}}(s)^*. \tag{1.3}$$

We here emphasize that $U(t,s)$ enjoys the domain invariance property

$$U(t,s)\mathscr{D}((p^2+x^2)^n) \subset \mathscr{D}((p^2+x^2)^n), \quad n \in \boldsymbol{N}, \tag{1.4}$$

and that $U(t, s)$ is strongly continuous in $\mathscr{D}((p^2+x^2)^n)$ with respect to (t, s) under the assumptions $(V)_{c,L}$, and $(V)_{\bar{c},G}$ or $(V)_{\bar{c},D,\rho}$ (see [A4] for the details).

We now note that for $a \subset c$, $H^a(t)$ is independent of time t because of $E^a(t) \equiv 0$. Thus we write it as H^a. Then $U_a(t, s)$ is written as

$$U_a(t,s) = e^{-i(t-s)H^a} \otimes \bar{U}_a(t,s). \tag{1.5}$$

We here introduce

$$U_{a,D}(t,0) = U_a(t,0)e^{-i\int_0^t I_a^c(p_a\tau)\,d\tau} \tag{1.6}$$

for $a \subset c$. Here $I_a^c = I_a - I_c$ and $p_a = -i\nabla_a$ is the velocity operator on $L^2(X_a)$. Under the assumptions $(V)_{c,L}$ and $(V)_{\bar{c},G}$, we define the modified wave operators $W_{a,G}^{D,\pm}$, $a \subset c$, by

$$W_{a,G}^{D,\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_{a,D}(t,0) e^{-i\int_0^t I_c(\bar{c}(\tau))\,d\tau}(P^a \otimes \mathrm{Id}), \tag{1.7}$$

where $P^a : L^2(X^a) \to L^2(X^a)$ is the eigenprojection associated with H^a. We call $e^{-i\int_0^t I_c(\bar{c}(\tau))\,d\tau}$ the Graf (or Zorbas)-type modifier (see [A1], [AT1], [Gr3], [HMS2] and [Zo]).

One of the main results of this article is the following theorem:

Theorem 1.1. *Assume that $(V)_{c,L}$ and $(V)_{\bar{c},G}$ are fulfilled. Then the modified wave operators $W_{a,G}^{D,\pm}$, $a \subset c$, exist, and are asymptotically complete*

$$L^2(X) = \sum_{a\subset c} \oplus \operatorname{Ran} W_{a,G}^{D,\pm}.$$

Next we suppose that $(V)_{\bar{c},D,\rho}$ with $0 < \rho \le 1/2$ instead of $(V)_{\bar{c},G}$ is satisfied. First we consider the case where $c \ne a_{\min}$, that is, $\#(c) \ne N$. Since $2 \le \#(c) < N$ by assumption, $N \ge 3$ is assumed here. Under the assumptions $(V)_{c,L}$ and $(V)_{\bar{c},D,\rho}$ with $(\sqrt{3}-1)/2 < \rho \le 1/2$, we define the modified wave operators $W_{a,D}^{D,\pm}$, $a \subset c$, by

$$W_{a,D}^{D,\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_{a,D}(t,0) e^{-i\int_0^t I_c(p_c\tau+\bar{c}(\tau))\,d\tau}(P^a \otimes \mathrm{Id}). \tag{1.8}$$

Then we have the following theorem:

Theorem 1.2. *Assume that $c \ne a_{\min}$ and that $(V)_{c,L}$ and $(V)_{\bar{c},D,\rho}$ with $(\sqrt{3}-1)/2 < \rho \le 1/2$ are fulfilled. Then the modified wave operators $W_{a,D}^{D,\pm}$, $a \subset c$, exist, and are asymptotically complete*

$$L^2(X) = \sum_{a\subset c} \oplus \operatorname{Ran} W_{a,D}^{D,\pm}.$$

Finally, we consider the case where $c = a_{\min}$. For example, when $N = 2$, $c = a_{\min}$ is satisfied by assumption. We here note that if $c = a_{\min}$,

$$H_c(t) = -\frac{1}{2}\Delta - E(t)\cdot x \equiv H_0(t),$$

$I_c(x) = V(x)$, $x_c = x$ and $p_c = p$, where $p = -i\nabla$ is the velocity operator on $L^2(X)$. $U_0(t,s)$ denotes the unitary propagator generated by $H_0(t)$. Under the assumption $(V)_{\bar{c},D,\rho}$ with $0 < \rho \le 1/2$, an approximate solution of the Hamilton-Jacobi equation

$$(\partial_t K)(t,\xi) = \frac{1}{2}(\xi + \tilde{b}(t))^2 + V((\nabla_\xi K)(t,\xi))$$

can be constructed (see [A4]). If $V \equiv 0$ and $K(0,\xi) \equiv 0$, $K(t,\xi)$ is written as

$$K(t,\xi) = K_0(t,\xi) \equiv \frac{t}{2}\xi^2 + \tilde{c}(t)\cdot\xi + \tilde{a}(t), \tag{1.9}$$

where $\tilde{a}(t)$ and $\tilde{c}(t)$ are as in (1.2). We here note that $(\nabla_\xi K_0)(t,\xi)$ is written as

$$(\nabla_\xi K_0)(t,\xi) = \xi t + \tilde{c}(t). \tag{1.10}$$

Under the assumptions $c = a_{\min}$ and $(V)_{\bar{c},D,\rho}$ with $0 < \rho \le 1/2$, we define the modified wave operators $W^\pm_{0,D}$ by

$$W^\pm_{0,D} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) e^{-i\int_T^t V((\nabla_\xi K)(\tau,p))\,d\tau}. \tag{1.11}$$

If $1/4 < \rho \le 1/2$, $e^{-i\int_T^t V((\nabla_\xi K)(\tau,p))\,d\tau}$ in (1.11) can be replaced by $e^{-i\int_0^t V((\nabla_\xi K_0)(\tau,p))\,d\tau} = e^{-i\int_0^t V(p\tau+\tilde{c}(\tau))\,d\tau}$, which is called the Dollard-type modifier (see [A1], [AT2], [JO], [JY] and [W]).

Then we have the following theorem:

Theorem 1.3. *Assume that $c = a_{\min}$ and $(V)_{\bar{c},D,\rho}$ with $0 < \rho \le 1/2$ are fulfilled. Then the modified wave operators $W^\pm_{0,D}$ exist and are unitary on $L^2(X)$.*

Remark 1.1. In our analysis, we need a certain regularity of V_{jk} like being at least in $C_b^8(\boldsymbol{R}^d)$ in order to obtain some propagation estimates which are useful for proving the asymptotic completeness of wave operators (see §3, in particular Lemma 3.6).

The initial time 0 can be replaced by any $s \in \boldsymbol{R}$.

For time-dependent Hamiltonians, the lack of energy conservation is a barrier in studying this problem. For instance, the time-boundedness of the kinetic energy was the key fact for studying the charge transfer model (see e.g. [Gr1]). Howland [Ho1] proposed the stationary scattering theory for time-dependent Hamiltonians, whose formulation was the quantum analogue to the procedure in the classical mechanics in order to 'recover' the conservation of energy. Yajima [Ya1] applied this Howland method to the two-body quantum systems with time-periodic short-range potentials and studied the problem of the asymptotic completeness for the systems (see also [Ho2] and [Yo1]). His result was extended to the three-body case by Nakamura [N] later (as for the spectral theory for general N-body systems, see Møller-Skibsted [MøS]). Under the same assumption on $\mathscr{E}(t)$ as in this article, Møller [Mø] studied the scattering theory for two-body quantum systems with short-range interactions, and Adachi [A3] also studied the scattering theory for N-body quantum systems with short-range interactions between particles whose specific charges are different as mentioned before, by using the so-called Howland-Yajima method.

The Howland-Yajima method reduces the problem under consideration to the problem of the asymptotic completeness of the usual wave operators associated with the Floquet Hamiltonian given by $K = -i\partial_t + H(t)$ on $L^2(\boldsymbol{T}; L^2(X))$ formally. Thus this method matches the quantum scattering theory for time-periodic short-range interactions, but seems not sufficient for the time-periodic long-range ones. For instance, Kitada-Yajima [KY] dealt with the so-called AC Stark effect, in which the mean of $\mathscr{E}(t)$ in t is *zero*, for two-body quantum systems with long-range interactions, by using the so-called Enss method. As implied by this, in studying the scattering theory for time-periodic long-range interactions, one needs to know some propagation properties of the physical propagator $U(t,s)$. One of purposes of this article is to give some propagation estimates for $U(t,s)$ (see §3), that was not done in [Mø] and [A3]. In the case where $\mathscr{E}(t) = \mathscr{E} + o(1)$, which is *not* time-periodic, this was done by Yokoyama [Yo2] for two-body systems with short-range interactions.

In the argument below, we will consider the case where $t \to \infty$ only. The case where $t \to -\infty$ can be dealt with quite similarly. For an X-valued operator L, $(L^2)^{1/2}$ is denoted by $|L|$ for brevity's sake.

2 Asymptotic clustering

In this section, we prove the so-called asymptotic clustering for the system under consideration, which is the key to showing Theorems 1.1, 1.2 and 1.3. Throughout this and the next sections, we suppose that $(V)_{c,L}$ and

$(V)'_{\tilde{c},D,\rho}$ $V_{jk}(r) \in C^\infty(\boldsymbol{R}^d)$, $(j,k) \not\subset c$, is a real-valued function and satisfies

$$|\partial^\beta V_{jk}(r)| \le C_\beta \langle r\rangle^{-(\rho+|\beta|/2)}, \quad |\beta| \le 1,$$
$$|\partial^\beta V_{jk}(r)| \le C_\beta, \quad |\beta| \ge 2,$$

with $0 < \rho \le 1/2$ are fulfilled. We note that under $(V)_{\tilde{c},G}$ with $\rho = \rho_G$ or $(V)_{\tilde{c},D,\rho}$, $(V)'_{\tilde{c},D,\rho}$ is fulfilled.

In this article, we often use the following convention for smooth cut-off functions F with $0 \le F \le 1$: For sufficiently small $\delta > 0$, we define

$$F(s \le d) = 1 \quad \text{for} \quad s \le d - \delta, \quad = 0 \quad \text{for} \quad s \ge d,$$
$$F(s \ge d) = 1 \quad \text{for} \quad s \ge d + \delta, \quad = 0 \quad \text{for} \quad s \le d,$$

and $F(d_1 \le s \le d_2) = F(s \ge d_1)\, F(s \le d_2)$. To clarify the dependence on $\delta > 0$ in the definition of F, we often write F_δ for F.

We now introduce the time-dependent intercluster potential $I_c(t,x)$ as

$$I_c(t,x) = I_c(x) F_{\varepsilon_1}(t^{-2}|x - \tilde{c}(t)| \le 2\varepsilon_1) \tag{2.1}$$

with some sufficiently small $\varepsilon_1 > 0$, where $\tilde{c}(t)$ is defined by (1.2). Since

$$\tilde{c}(t) - \frac{E}{2}t^2 = \int_0^t (\tilde{b}(s) - Es)\, ds = O(t) \tag{2.2}$$

in virtue of the periodicity of $\tilde{b}(t) - Et$ by the definition of E, we see that $I_c(t,x)$ enjoys the estimate

$$|\partial_x^\beta I_c(t,x)| \le C_\beta(t + \langle x\rangle^{1/2})^{-(2\rho+|\beta|)}, \quad |\beta| \le 1, \tag{2.3}$$

for $t > 0$, if $0 < \varepsilon_1 < \min_{\alpha \not\subset c} |E^\alpha|/4$. Then we define the time-dependent Hamiltonian $\tilde{H}_c(t)$ by

$$\tilde{H}_c(t) = H_c(t) + I_c(t,x), \tag{2.4}$$

and denote by $\tilde{U}_c(t)$, $t \ge T$, the unitary propagator generated by $\tilde{H}_c(t)$ such that $\tilde{U}_c(T) = \mathrm{Id}$. We here note that the domain invariance property of $\tilde{U}_c(t)$

$$\tilde{U}_c(t)\mathscr{D}(p^2 + x^2) \subset \mathscr{D}(p^2 + x^2)$$

holds and that $\tilde{U}_c(t)$ is strongly continuous in $\mathscr{D}(p^2 + x^2)$ with respect to t.

In order to prove Theorems 1.1, 1.2 and 1.3, we will claim that the following asymptotic clustering holds:

Theorem 2.1 (Asymptotic Clustering). *Assume that $(V)_{c,L}$ and $(V)'_{\tilde{c},D,\rho}$ with $0 < \rho \le 1/2$ are fulfilled. Then the strong limit*

$$\tilde{\Omega}_c = \operatorname*{s-lim}_{t\to\infty} U(t,0)^*\tilde{U}_c(t) \tag{2.5}$$

exists and is unitary on $L^2(X)$.

This property played an important role to prove the asymptotic completeness of N-body quantum systems in a (time-independent or time-periodic) homogeneous electric field in the works of Adachi and Tamura [AT1, AT2], and Adachi [A3] (see also [A1] and [HMS2]).

In order to prove Theorem 2.1, we need the following propagation estimates for both $\tilde{U}_c(t)$ and $U(t,0)$. From now on the norm and scalar product in a Hilbert space $\mathscr{H}_1$ are denoted by $\|\cdot\|_{\mathscr{H}_1}$ and $(\cdot,\cdot)_{\mathscr{H}_1}$, respectively. The norm of bounded operators on $\mathscr{H}_1$ is also denoted by $\|\cdot\|_{\mathscr{B}(\mathscr{H}_1)}$:

Proposition 2.2. *The following estimates hold for $\phi \in \mathscr{D}(p^2 + x^2)$ as $t \to \infty$:*

$$\big\| |p - \tilde{b}(t)| \tilde{U}_c(t)\phi \big\|_{L^2(X)} = O(1), \tag{2.6}$$

$$\big\| |x - \tilde{c}(t)| \tilde{U}_c(t)\phi \big\|_{L^2(X)} = O(t). \tag{2.7}$$

Corollary 2.3. *Let $\varepsilon > 0$. Then the following estimate holds for $\phi \in \mathscr{D}(p^2 + x^2)$ as $t \to \infty$:*

$$\big\| F_\varepsilon(t^{-2}|x - \tilde{c}(t)| \ge \varepsilon)\tilde{U}_c(t)\phi \big\|_{L^2(X)} = O(t^{-1}). \tag{2.8}$$

These can be shown by computing the Heisenberg derivatives of H^c, $p_c - \tilde{b}(t)$ and $x - \tilde{c}(t)$ associated with $\tilde{H}_c(t)$. Here the Heisenberg derivative of $\Phi(t)$ associated with $H(t)$ is denoted by

$$\mathbf{D}_{H(t)}(\Phi(t)) = \frac{\partial \Phi}{\partial t}(t) + i[H(t), \Phi(t)].$$

For the details, see [A4].

Theorem 2.4. *Let* $0 < \varepsilon < \min_{\alpha \not\subset c} |E^\alpha|/4$. *Then the following estimates hold for* $\phi \in \mathscr{D}((p^2 + x^2)^2)$ *as* $t \to \infty$*:*

$$\left\| F_\varepsilon(t^{-2}|x - \tilde{c}(t)| \geq \varepsilon) U(t,0)\phi \right\|_{L^2(X)} = O(t^{-1/2}), \tag{2.9}$$

$$\left\| |p - \tilde{b}(t)| F_\varepsilon(t^{-2}|x - \tilde{c}(t)| \leq 2\varepsilon) U(t,0)\phi \right\|_{L^2(X)} = O(t^{1/2}), \tag{2.10}$$

$$\left\| |x - \tilde{c}(t)| F_\varepsilon(t^{-2}|x - \tilde{c}(t)| \leq 2\varepsilon) U(t,0)\phi \right\|_{L^2(X)} = O(t^{3/2}). \tag{2.11}$$

Theorem 2.4 is one of the main results of this article. In the next section, we describe the outline of the proof. We will now prove Theorem 2.1 under the assumption that Theorem 2.4 holds.

Proof of Theorem 2.1. We have only to prove the existence of the limits

$$\lim_{t\to\infty} U(t,0)^* \tilde{U}_c(t)\phi, \quad \lim_{t\to\infty} \tilde{U}_c(t)^* U(t,0)\phi$$

for $\phi \in \mathscr{D}((p^2+x^2)^2)$, because $\mathscr{D}((p^2+x^2)^2)$ is dense in $L^2(X)$. We here put $\eta(t) = F_{\varepsilon_1/2}(t^{-2}|x - \tilde{c}(t)| \leq \varepsilon_1)$. By virtue of Corollary 2.3 and Theorem 2.4, we see that

$$\lim_{t\to\infty} U(t,0)^*(1-\eta(t))\tilde{U}_c(t)\phi = 0, \quad \lim_{t\to\infty} \tilde{U}_c(t)^*(1-\eta(t))U(t,0)\phi = 0.$$

Thus we have only to show the existence of the limits

$$\lim_{t\to\infty} U(t,0)^*\eta(t)\tilde{U}_c(t)\phi, \quad \lim_{t\to\infty} \tilde{U}_c(t)^*\eta(t)U(t,0)\phi. \tag{2.12}$$

We here note that

$$I_c(x)\eta(t) = I_c(t,x)\eta(t)$$

for $t > 0$, which is the key in the proof. Since

$$\begin{aligned}
&\frac{d}{dt}(U(t,0)^*\eta(t)\tilde{U}_c(t)\phi) \\
&= U(t,0)^*[\eta_1(t)\cdot\{-2t^{-3}(x-\tilde{c}(t)) + t^{-2}(p - \tilde{b}(t))\} + O(t^{-4})]\tilde{U}_c(t)\phi, \\
&\frac{d}{dt}(\tilde{U}_c(t)^*\eta(t)U(t,0)\phi) \\
&= \tilde{U}_c(t)^*[\{-2t^{-3}(x-\tilde{c}(t)) + t^{-2}(p-\tilde{b}(t))\}\cdot\eta_1(t) + O(t^{-4})]U(t,0)\phi
\end{aligned}$$

with $\eta_1(t) = F'_{\varepsilon_1/2}(t^{-2}|x-\tilde{c}(t)| \leq \varepsilon_1)(x - \tilde{c}(t))/|x-\tilde{c}(t)|$, we obtain from Proposition 2.2 and Theorem 2.4

$$\begin{aligned}
\left\| \frac{d}{dt}(U(t,0)^*\eta(t)\tilde{U}_c(t)\phi) \right\|_{L^2(X)} &= O(t^{-2}), \\
\left\| \frac{d}{dt}(\tilde{U}_c(t)^*\eta(t)U(t,0)\phi) \right\|_{L^2(X)} &= O(t^{-3/2}),
\end{aligned}$$

which implies the existence of (2.12) by virtue of the Cook-Kuroda method. Thus the proof is completed. □

Remark 2.1. If $\rho > 1/2$, that is, if all V_{jk}'s with $(j,k) \not\subset c$ are Stark short-range,

$$\operatorname*{s-lim}_{t\to\infty} \tilde{U}_c(t)^* U_c(t,0)$$

exists and is unitary on $L^2(X)$, by virtue of (2.3) with $-2\rho < -1$. Therefore it follows from this and Theorem 2.1 that

$$\Omega_c = \operatorname*{s-lim}_{t\to\infty} U(t,0)^* U_c(t,0) \tag{2.13}$$

exists and unitary on $L^2(X)$. This gives an alternative proof of the asymptotic completeness obtained in Møller [Mø] and Adachi [A3].

3 Propagation estimates for $U(t,0)$

We first move the oscillation arising from $E(t) - E$ into the potential V, and reduce the present problem to the one for a so-called N-body Stark Hamiltonian with a certain time-periodic potential, by using a version of the Avron-Herbst formula initiated by Møller [Mø]: We define T-periodic functions on $\boldsymbol{R}$

$$\begin{aligned}
b(t) &= \int_0^t (E(s) - E)\,ds - b_0, \quad b_0 = \frac{1}{T}\int_0^T \int_0^t (E(s) - E)\,dsdt, \\
c(t) &= \int_0^t b(s)\,ds - c_0, \quad c_0 = \frac{1}{T}\int_0^T \left(-\frac{1}{2}|b(t)|^2 + \int_0^t E\cdot b(s)\,ds\right) dt \frac{E}{|E|^2}, \\
a(t) &= \int_0^t \left(\frac{1}{2}|b(s)|^2 - E\cdot c(s)\right) ds,
\end{aligned} \tag{3.1}$$

where $b(t), c(t) \in X$ and $a(t) \in \boldsymbol{R}$, and a strongly continuous periodic family of unitary operators on $L^2(X)$ by

$$\mathscr{T}(t) = e^{-ia(t)} e^{ib(t)\cdot x} e^{-ic(t)\cdot p}. \tag{3.2}$$

We here note that the constants b_0 and c_0 in (3.1) are chosen in order to make $c(t)$ and $a(t)$ T-periodic. Moreover we define the time-dependent Hamiltonian $H^S(t)$ on $L^2(X)$ by

$$H^S(t) = H_0^S + V(x + c(t)), \quad H_0^S = -\frac{1}{2}\Delta - E\cdot x. \tag{3.3}$$

H_0^S is called the free Stark Hamiltonian. We note that the time-periodic potential $V(x + c(t))$ is written as

$$V(x + c(t)) = V^c(x) + I_c(x + c(t)), \tag{3.4}$$

because $c(t) \in X_c$ by definition and $V^c(x) = V^c(x^c)$ is independent of $x_c \in X_c$ also by definition. Put

$$b^S(t) = \int_0^t E\,d\tau = Et, \quad c^S(t) = \int_0^t b^S(\tau)\,d\tau = \frac{E}{2}t^2, \tag{3.5}$$

and define $\mathscr{T}^S(t)$ as

$$\mathscr{T}^S(t) = e^{-ia^S(t)}e^{ib^S(t)\cdot x}e^{-ic^S(t)\cdot p}, \quad a^S(t) = \frac{1}{2}\int_0^t b^S(\tau)^2\,d\tau. \tag{3.6}$$

It is well known that the original Avron-Herbst formula [AH] holds:

$$e^{-itH_0^S} = \mathscr{T}^S(t)e^{-itH_0^{Sc}}, \quad H_0^{Sc} = -\frac{1}{2}\Delta \tag{3.7}$$

Let $U^S(t,s)$ be the unitary propagator generated by the Hamiltonian $H^S(t)$, whose existence and uniqueness can be guaranteed by the Avron-Herbst formula

$$U(t,s) = \mathscr{T}(t)U^S(t,s)\mathscr{T}(s)^*, \quad \text{or} \quad U^S(t,s) = \mathscr{T}^S(t)U^{Sc}(t,s)\mathscr{T}^S(s)^*. \tag{3.8}$$

We here note that the domain invariance property of $U^S(t,0)$

$$U^S(t,0)\mathscr{D}((p^2+x^2)^n) \subset \mathscr{D}((p^2+x^2)^n), \quad n \in \boldsymbol{N},$$

holds and that $U^S(t,0)$ is strongly continuous in $\mathscr{D}((p^2+x^2)^n)$ with respect to t, by virtue of the property of $U(t,s)$ mentioned in §1. Noting that

$$\begin{aligned}
\mathscr{T}(t)^*(p-\tilde{b}(t))\mathscr{T}(t) &= p - \tilde{b}(t) + b(t) = p - b^S(t) - b_0, \\
\mathscr{T}(t)^*(x-\tilde{c}(t))\mathscr{T}(t) &= x - \tilde{c}(t) + c(t) = x - c^S(t) - (b_0t + c_0)
\end{aligned}$$

by virtue of (3.1), we see that Theorem 2.4 is equivalent to the following:

Theorem 3.1. *Let* $0 < \varepsilon < \min_{\alpha \not\subset c}|E^\alpha|/4$. *Then the following estimates hold for* $\phi \in \mathscr{D}((p^2+x^2)^2)$ *as* $t \to \infty$:

$$\left\|F_\varepsilon(t^{-2}|x-c^S(t)| \geq \varepsilon)U^S(t,0)\phi\right\|_{L^2(X)} = O(t^{-1/2}), \tag{3.9}$$

$$\left\||p-b^S(t)|F_\varepsilon(t^{-2}|x-c^S(t)| \leq 2\varepsilon)U^S(t,0)\phi\right\|_{L^2(X)} = O(t^{1/2}), \tag{3.10}$$

$$\left\||x-c^S(t)|F_\varepsilon(t^{-2}|x-c^S(t)| \leq 2\varepsilon)U^S(t,0)\phi\right\|_{L^2(X)} = O(t^{3/2}). \tag{3.11}$$

Now we introduce the Floquet Hamiltonian associated with $H^S(t)$, which is key in the Howland-Yajima method (see Howland [Ho1, Ho2] and Yajima [Ya1]). We let $\boldsymbol{T} = \boldsymbol{R}/(T\boldsymbol{Z})$ be the torus and introduce $\mathscr{H} = L^2(\boldsymbol{T}; L^2(X)) \cong L^2(\boldsymbol{T})\otimes L^2(X)$. We define a family of operators $\{\hat{U}(\sigma)\}_{\sigma\in\boldsymbol{R}}$ on $\mathscr{H}$ by

$$(\hat{U}(\sigma)f)(t) = U^S(t, t-\sigma)f(t-\sigma) \tag{3.12}$$

for $f \in \mathscr{H}$. Since $\{\hat{U}(\sigma)\}_{\sigma\in\boldsymbol{R}}$ forms a strongly continuous unitary group on $\mathscr{H}$, $\hat{U}(\sigma)$ is written as

$$\hat{U}(\sigma) = e^{-i\sigma K}, \tag{3.13}$$

where $K = D_t + H^S(t)$ is a self-adjoint operator on $\mathscr{H}$, where $D_t = -i\partial_t$ is a self-adjoint operator on $L^2(\boldsymbol{T})$ with its domain $AC^2(\boldsymbol{T})$, which is the space of absolutely continuous functions on $\boldsymbol{T}$ with their derivatives being square integrable (following the notation in [RS]). K is called the Floquet Hamiltonian associated with $H^S(t)$.

The following two theorems show some spectral properties of K, which can be proved in the same way as in [A3] (see also Herbst-Møller-Skibsted [HMS1]) by using

$$|V_{jk}(r)| + |\nabla V_{jk}(r)| = o(1)$$

as $|r| \to \infty$, which is fulfilled under $(V)_{c,L}$ and $(V)'_{\bar{c},D,\rho}$ with $0 < \rho \leq 1/2$. So we omit the proof.

Theorem 3.2 (Absence of Bound States). *The pure point spectrum $\sigma_{pp}(K)$ of the Floquet Hamiltonian K is empty.*

Theorem 3.3 (Mourre Estimate). *Let $A = E \cdot p/|E|$ and $0 < \nu < |E| < \nu'$. Then one can take $\delta > 0$ so small uniformly in $\lambda \in \boldsymbol{R}$ that*

$$\eta_\delta(K-\lambda)i[K,A]\eta_\delta(K-\lambda) \geq \nu\eta_\delta(K-\lambda)^2, \tag{3.14}$$

$$\eta_\delta(K-\lambda)i[K,-A]\eta_\delta(K-\lambda) \geq -\nu'\eta_\delta(K-\lambda)^2 \tag{3.15}$$

hold, where $\eta_\delta \in C_0^\infty(\boldsymbol{R})$ satisfies $0 \leq \eta_\delta \leq 1$, $\eta_\delta(t) = 1$ for $|t| \leq \delta$ and $\eta_\delta(t) = 0$ for $|t| \geq 2\delta$. In particular, the spectrum of K is purely absolutely continuous.

Now we prepare the maximal and minimal acceleration bounds for $e^{-i\sigma K}$, by following the abstract theory of Skibsted [Sk]. For the proofs, see [A4].

Proposition 3.4 (Maximal Acceleration Bound). *Let $f \in C_0^\infty(\boldsymbol{R})$, $s_0 \geq s_1 \geq 0$, and $\varepsilon > 0$. Then there exists $M > 0$ such that the following estimate holds as $\sigma \to \infty$:*

$$\|(\sigma^{-1}\langle p\rangle)^{s_1}F_\varepsilon(\sigma^{-1}\langle p\rangle \geq M)e^{-i\sigma K}f(K)\langle p\rangle^{-s_0}\|_{\mathscr{B}(\mathscr{H})} = O(\sigma^{-s_0}). \tag{3.16}$$

Proposition 3.5 (Minimal Acceleration Bound). *Let $f \in C_0^\infty(\boldsymbol{R})$, $s_0 \geq s_1 \geq 0$ and $\varepsilon > 0$. Let A, ν and ν' be as in Theorem 3.3. Then the following estimates hold as $\sigma \to \infty$:*

$$\|(\nu - \sigma^{-1}A)^{s_1}F_\varepsilon(\sigma^{-1}A \leq \nu - \varepsilon)e^{-i\sigma K}f(K)\langle A\rangle^{-s_0}\|_{\mathscr{B}(\mathscr{H})} = O(\sigma^{-s_0}), \tag{3.17}$$

$$\|(\sigma^{-1}A - \nu')^{s_1}F_\varepsilon(\sigma^{-1}A \geq \nu' + \varepsilon)e^{-i\sigma K}f(K)\langle A\rangle^{-s_0}\|_{\mathscr{B}(\mathscr{H})} = O(\sigma^{-s_0}). \tag{3.18}$$

In order to translate these propagation estimates for $e^{-i\sigma K}$ into the ones for $U^S(t,0)$, we need the following lemma.

Lemma 3.6. *Let $f \in C_0^\infty(\boldsymbol{R})$, $s_0 \geq s_1 \geq 0$, and $\varepsilon > 0$. Let A, ν and ν' be as in Theorem 3.3. Let M be as in Proposition 3.4. Let J_{σ,s_1} be one of the following three operators on $\mathscr{H}$:*

$$(\sigma^{-1}\langle p\rangle)^{s_1}F_\varepsilon(\sigma^{-1}\langle p\rangle \geq M), \quad (\nu - \sigma^{-1}A)^{s_1}F_\varepsilon(\sigma^{-1}A \leq \nu - \varepsilon),$$
$$(\sigma^{-1}A - \nu')^{s_1}F_\varepsilon(\sigma^{-1}A \geq \nu' + \varepsilon).$$

Then the following estimate holds as $\sigma \to \infty$:

$$\|\langle D_t\rangle J_{\sigma,s_1}e^{-i\sigma K}f(K)\langle p\rangle^{-s_0}\langle D_t\rangle^{-1}\|_{\mathscr{B}(\mathscr{H})} = O(\sigma^{1-s_0}). \tag{3.19}$$

Proof. Since

$$
\begin{aligned}
-i\mathrm{ad}_{D_t}(K) &= \nabla I_c(x + c(t)) \cdot b(t), \\
(-i)^2\mathrm{ad}^2_{D_t}(K) &= \nabla I_c(x + c(t)) \cdot (E(t) - E) + b(t)^* \nabla^2 I_c(x + c(t)) b(t),
\end{aligned}
$$

are bounded on $\mathscr{H}$, it can be shown easily that

$$
\langle D_t\rangle^2 e^{-i\sigma K} f(K) \langle D_t\rangle^{-2} = O(\sigma^2),
$$

which implies

$$
\langle D_t\rangle^2 J_{\sigma,0} e^{-i\sigma K} f(K) \langle D_t\rangle^{-2} = O(\sigma^2)
$$

because p does commute with D_t. Noting that p does commute with D_t again, by complex interpolation between this and

$$
J_{\sigma,2s_1} e^{-i\sigma K} f(K) \langle p\rangle^{-2s_0} = O(\sigma^{-2s_0})
$$

in virtue of Hadamard's three line theorem, we obtain (3.19). □

Now we will translate the obtained propagation estimates for $e^{-i\sigma K}$ into the ones for $U^S(t, 0)$. Take $s_0 = 2$. Let $\phi \in \mathscr{D}((p^2 + x^2)^2) \subset L^2(X)$ and put $\phi(t) = U^S(t, 0)\phi$. Then we see that $\phi(t) \in \mathscr{D}(D_t)$ and that $D_t\phi(t) \in \mathscr{D}(p^2 + x^2)$ by virtue of the domain invariance property of $U^S(t, 0)$ mentioned before. Let $\mathscr{U}$ be the unitary operator on $\mathscr{H}$ defined by

$$
(\mathscr{U}\psi)(t) = U^S(t, 0)\psi(t), \quad t \in \boldsymbol{T},\ \psi(t) \in \mathscr{H}.
$$

It is known that

$$
e^{-iTK} = \mathscr{U}(\mathrm{Id} \otimes U^S(T, 0))\mathscr{U}^* \tag{3.20}
$$

holds on $\mathscr{H} \cong L^2(\boldsymbol{T}) \otimes L^2(X)$ (see Yajima-Kitada [YK]). Then we have

$$
(f(K)\phi)(t) = U^S(t, 0) g(U^S(T, 0))\phi, \quad t \in \boldsymbol{T},
$$

where $f \in C_0^\infty(\boldsymbol{R})$ supported in $(\lambda_0 - \pi/T, \lambda_0 + \pi/T)$ for some $\lambda_0 \in \boldsymbol{R}$, and g is the function on the unit-circle defined by $g(e^{-iT\lambda}) = f(\lambda)$ (see Møller-Skibsted [MøS]). We here note the following: Let $J = J(t)$ be an operator on $\mathscr{H}$, and $\psi = \psi(t) \in \mathscr{H}$ be such that $e^{-i\sigma K}\psi \in \mathscr{D}(J)$. Then

$$
\|Je^{-i\sigma K}\psi\|^2_{\mathscr{H}} = \int_0^T \|J(t + \sigma)U^S(t + \sigma, t)\psi(t)\|^2_{L^2(X)}\, dt
$$

holds. Noting that J_{σ,s_1} in Lemma 3.6 is independent of t, we see that

$$
\begin{aligned}
&\int_0^T \|J_{\sigma,s_1} U^S(t + \sigma, 0) g(U^S(T, 0))\phi\|^2_{L^2(X)}\, dt = O(\sigma^{-2}), \\
&\int_0^T \|\partial_t\{J_{\sigma,s_1} U^S(t + \sigma, 0) g(U^S(T, 0))\phi\}\|^2_{L^2(X)}\, dt = O(\sigma^{-2})
\end{aligned}
$$

hold with $0 \le s_1 \le 2$, by virtue of the above formula and Lemma 3.6. From these, we obtain

$$\|J_{\sigma,s_1}U^S(t+\sigma,0)g(U^S(T,0))\phi\|^2_{L^2(X)} \in W^{1,1}(0,T),$$
$$\big\| \|J_{\sigma,s_1}U^S(t+\sigma,0)g(U^S(T,0))\phi\|^2_{L^2(X)} \big\|_{W^{1,1}(0,T)} = O(\sigma^{-2})$$

by the Schwarz inequality. Here $W^{1,1}(0,T) = \{u \in L^1(0,T) \mid u' \in L^1(0,T)\}$ is a Sobolev space on the interval $(0,T)$. By using the Sobolev imbedding theorem (see e.g. [B]), we obtain

$$\big\| \|J_{\sigma,s_1}U^S(t+\sigma,0)g(U^S(T,0))\phi\|^2_{L^2(X)} \big\|_{L^\infty(0,T)} = O(\sigma^{-2}),$$

which implies

$$\|J_{\sigma,s_1}U^S(\sigma,0)g(U^S(T,0))\phi\|_{L^2(X)} = O(\sigma^{-1}).$$

Therefore the following propagation estimates can be obtained by using a partition of unity on the unit-circle.

Proposition 3.7. *Let $0 \le s_1 \le 2$ and $\varepsilon > 0$. Let A, ν and ν' be as in Theorem 3.3. Let M be as in Proposition 3.4. Then the following estimates hold for $\phi \in \mathscr{D}((p^2+x^2)^2)$ as $t \to \infty$:*

$$\big\|(t^{-1}\langle p\rangle)^{s_1}F_\varepsilon(t^{-1}\langle p\rangle \ge M)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1}), \tag{3.21}$$
$$\big\|(\nu - t^{-1}A)^{s_1}F_\varepsilon(t^{-1}A \le \nu - \varepsilon)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1}), \tag{3.22}$$
$$\big\|(t^{-1}A - \nu')^{s_1}F_\varepsilon(t^{-1}A \ge \nu' + \varepsilon)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1}). \tag{3.23}$$

Based on these estimates, we will derive some useful propagation estimates for $U^S(t,0)$. For the proofs, see [A4].

Proposition 3.8 (Maximal Acceleration Bound). *Let $0 \le s_1 \le 1/2$ and $\varepsilon > 0$. Then there exists $M' > 0$ such that following estimate holds for $\phi \in \mathscr{D}((p^2+x^2)^2)$ as $t \to \infty$:*

$$\big\|(t^{-2}\langle x\rangle)^{s_1}F_\varepsilon(t^{-2}\langle x\rangle \ge M')U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1}). \tag{3.24}$$

Proposition 3.9 (Minimal Acceleration Bound). *Let $0 \le s_1 \le 1/2$ and $\varepsilon > 0$. Let ν and ν' be as in Theorem 3.3. Then the following estimates hold for $\phi \in \mathscr{D}((p^2+x^2)^2)$ as $t \to \infty$:*

$$\big\|(\nu/2 - t^{-2}z)^{s_1}F_\varepsilon(t^{-2}z \le \nu/2 - \varepsilon)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1}), \tag{3.25}$$
$$\big\|(t^{-2}z - \nu'/2)^{s_1}F_\varepsilon(t^{-2}z \ge \nu'/2 + \varepsilon)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1}), \tag{3.26}$$

where $z = E \cdot x/|E|$.

Theorem 3.10. *Let $\varepsilon > 0$. Then the following estimates hold for $\phi \in \mathscr{D}((p^2+x^2)^2)$ as $t \to \infty$:*

$$\big\|F_\varepsilon(t^{-1}|p - b^S(t)| \ge \varepsilon)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{-1/2}). \tag{3.27}$$
$$\big\||p - b^S(t)|F_\varepsilon(t^{-1}|p - b^S(t)| \ge \varepsilon)U^S(t,0)\phi\big\|_{L^2(X)} = O(t^{1/2}). \tag{3.28}$$

By virtue of these estimates, one can show Theorem 3.1 in the same way as in [A2]. For the details, see [A4].

4 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Throughout this section, we assume that $(V)_{c,L}$ and $(V)_{\tilde{c},G}$ are fulfilled. We first note that under $(V)_{\tilde{c},G}$,

$$|\partial_x^\beta I_c(t,x)| \leq C_\beta (t + \langle x\rangle^{1/2})^{-2(\rho_G+|\beta|)}, \quad |\beta| \leq 1, \tag{4.1}$$

holds for $t > 0$, which is finer than (2.3).

We introduce the time-dependent Hamiltonian $H_{cG}(t)$ as

$$H_{cG}(t) = H_c(t) + I_c(\tilde{c}(t)). \tag{4.2}$$

$U_{cG}(t)$ denotes the propagator generated by $H_{cG}(t)$ such that $U_{cG}(0) = \mathrm{Id}$. We here note that

$$I_c(\tilde{c}(t)) = I_c(t, \tilde{c}(t)) \tag{4.3}$$

for $t > 0$, and that $U_{cG}(t)$ is represented as

$$U_{cG}(t) = U_c(t,0)e^{-i\int_0^t I_c(\tilde{c}(\tau))\,d\tau}. \tag{4.4}$$

Noticing $\mathbf{D}_{H_{cG}(t)}(H^c) = 0$, $\mathbf{D}_{H_{cG}(t)}(p_c - \tilde{b}(t)) = 0$ and $\mathbf{D}_{H_{cG}(t)}(x - \tilde{c}(t)) = p - \tilde{b}(t)$, the following propagation property of $U_{cG}(t)$ can be proved as in the proof of Proposition 2.2. We omit the proof.

Lemma 4.1. *The following estimate holds for $\phi \in \mathscr{D}(p^2 + x^2)$ as $t \to \infty$:*

$$\| |x - \tilde{c}(t)| U_{cG}(t)\phi \|_{L^2(X)} = O(t). \tag{4.5}$$

By using this lemma and Proposition 2.2, we obtain the following.

Proposition 4.2. *The strong limit*

$$\operatorname*{s-lim}_{t\to\infty} U_{cG}(t)^* \tilde{U}_c(t) \tag{4.6}$$

exists and is unitary on $L^2(X)$.

Proof. We have only to show the existence of

$$\lim_{t\to\infty} U_{cG}(t)^* \tilde{U}_c(t)\phi, \tag{4.7}$$

$$\lim_{t\to\infty} \tilde{U}_c(t)^* U_{cG}(t)\phi \tag{4.8}$$

for $\phi \in \mathscr{D}(p^2 + x^2)$. Using (4.3), we have

$$\frac{d}{dt}(U_{cG}(t)^* \tilde{U}_c(t)\phi) = U_{cG}(t)^* i(I_c(t,\tilde{c}(t)) - I_c(t,x))\tilde{U}_c(t)\phi.$$

Since

$$I_c(t,\tilde{c}(t)) - I_c(t,x) = -\int_0^1 (\nabla I_c)(t, sx + (1-s)\tilde{c}(t)) \cdot (x - \tilde{c}(t))\,ds$$

and $\sup_{x\in X} |(\nabla I_c)(t,x)| = O(t^{-2\rho_G-2})$ by $(V)_{\tilde{c},G}$, the existence of (4.7) can be proved by Proposition 2.2 and the Cook-Kuroda method, because $-2\rho_G - 2 + 1 < -1$. The existence of (4.8) can be proved quite similarly by virtue of (4.5). $\square$

Combining this with Theorem 2.1, we obtain the following, which is the key to the proof of Theorem 1.1:

Corollary 4.3. *The strong limit*

$$\Omega_{cG} = \operatorname*{s-lim}_{t\to\infty} U(t,0)^* U_{cG}(t) \tag{4.9}$$

exists and is unitary on $L^2(X)$.

Since

$$U_{cG}(t,0) = e^{-itH^c} \otimes (\bar{U}_c(t,0) e^{-i\int_0^t I_c(\tilde{c}(\tau))\, d\tau})$$

by (1.5), Theorem 1.1 can be proved in the same way as in [A3], [AT1] and [HMS2], by combining Corollary 4.3 and the following result of the asymptotic completeness for $H^c = -\Delta^c/2 + V^c(x^c)$, which is proved by Dereziński [D] (see also [DG1] and [Z]). So we omit the proofs: We introduce some notations. Suppose $a \subset c$. We define the cluster Hamiltonian $H_a^c = -\Delta^c/2 + V^a$ on $L^2(X^c)$ and put

$$U_{a,D}^c(t) = e^{-itH_a^c} e^{-i\int_0^t I_a^c(p_a u)\, du}$$

acting on $L^2(X^c)$. We put $X_a^c = X^c \ominus X^a$. Then we see that $L^2(X^c)$ is decomposed into $L^2(X^a) \otimes L^2(X_a^c)$. Thus H_a^c is decomposed into $H_a^c = H^a \otimes \mathrm{Id} + \mathrm{Id} \otimes T_a^c$ on $L^2(X^c) = L^2(X^a) \otimes L^2(X_a^c)$, where $T_a^c = -\Delta_a^c/2$ and Δ_a^c is the Laplace-Beltrami operator on X_a^c. It follows from this that

$$U_{a,D}^c(t) = e^{-itH^a} \otimes (e^{-itT_a^c} e^{-i\int_0^t I_a^c(p_a u)\, du}) \tag{4.10}$$

on $L^2(X^c) = L^2(X^a) \otimes L^2(X_a^c)$.

Theorem 4.4. *Assume that* $(V)_{c,L}$ *is fulfilled. Then the modified wave operators*

$$\Omega_a^{c,\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH^c} U_{a,D}^c(t)(P^a \otimes \mathrm{Id})$$

acting on $L^2(X^c)$, *exist for all* $a \subset c$, *and are asymptotically complete*

$$L^2(X^c) = \sum_{a\subset c} \oplus \operatorname{Ran} \Omega_a^{c,\pm}.$$

5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Throughout this section, we assume that $c \neq a_{\min}$ and that $(V)_{c,L}$ and $(V)_{\tilde{c},D,\rho}$ with $(\sqrt{3}-1)/2 < \rho \le 1/2$ are fulfilled. We first note that under $(V)_{\tilde{c},D,\rho}$,

$$|\partial_x^\beta I_c(t,x)| \le C_\beta (t + \langle x\rangle^{1/2})^{-(2\rho+|\beta|)}, \quad t > 0, \tag{5.1}$$

holds. Since the proof is quite similar to the one in Adachi-Tamura [AT2], we sketch it.

We introduce the time-dependent Hamiltonians

$$\tilde{H}_{aD}(t) = H_a(t) + I_a^c(p_a t) + I_c(p_a t + \tilde{c}(t) - t\tilde{b}(t)),$$

$$H_{a,1}(t) = H_a(t) + I_a^c(p_a t) + I_c(t, p_a t + \tilde{c}(t) - t\tilde{b}(t)),$$
$$H_c^{Sc}(t) = H_c^{Sc} + I_c(t, x + \tilde{c}(t)),$$
$$H_{a,1}^{Sc}(t) = H_a^{Sc} + I_a^c(p_a t) + I_c(t, p_a t + \tilde{c}(t))$$

for $a \subset c$, where $H_a^{Sc} = -\Delta/2 + V^a(x^a)$ acts on $L^2(X)$. $\tilde{U}_{aD}(t)$, $U_{a,1}(t)$, $U_c^{Sc}(t)$ and $U_{a,1}^{Sc}(t)$ denote the propagators generated by $\tilde{H}_{aD}(t)$, $H_{a,1}(t)$, $H_c^{Sc}(t)$ and $H_{a,1}^{Sc}(t)$, respectively, where $\tilde{U}_{aD}(0) = \mathrm{Id}$, $U_{a,1}(T) = \mathrm{Id}$, $U_c^{Sc}(T) = \mathrm{Id}$ and $U_{a,1}^{Sc}(T) = \mathrm{Id}$. Since $U_a(t,0)p_a U_a(t,0)^* = p_a - \tilde{b}(t)$ for $a \subset c$, $\tilde{U}_{aD}(t)$ is explicitly represented by

$$\tilde{U}_{aD}(t) = U_{a,D}(t,0)e^{-i\int_0^t I_c(p_a s + \tilde{c}(s))\, ds}.$$

Then the following Avron-Herbst formula holds:

$$\tilde{U}_c(t) = \tilde{\mathscr{T}}(t) U_c^{Sc}(t) \tilde{\mathscr{T}}(T)^*, \quad U_{a,1}(t) = \tilde{\mathscr{T}}(t) U_{a,1}^{Sc}(t) \tilde{\mathscr{T}}(T)^*. \tag{5.2}$$

By virtue of the relation (5.2), we have only to study the asymptotic behavior of $U_c^{Sc}(t)$. We now apply to $U_c^{Sc}(t)$the result by Dereziński [D] on the asymptotic completeness for long-range N-body quantum systems without electric fields.

Theorem 5.1. *Assume that $(V)_{c,L}$ and $(V)_{\tilde{c},D,\rho}$ with $(\sqrt{3}-1)/2 < \rho \leq 1/2$ are fulfilled. Then the modified wave operators*

$$\Omega_{a,1}^{Sc} = \operatorname*{s-lim}_{t\to\infty} U_c^{Sc}(t)^* U_{a,1}^{Sc}(t)(P^a \otimes \mathrm{Id})$$

exist for all $a \subset c$, and are asymptotically complete

$$L^2(X) = \sum_{a \subset c} \oplus \operatorname{Ran} \Omega_{a,1}^{Sc}.$$

The condition $2\rho > \sqrt{3} - 1$ is essentially used to prove this theorem only. By virtue of the Avron-Herbst formula (5.2), the following corollary is obtained as an immediate consequence of this theorem.

Corollary 5.2. *Assume that $(V)_{c,L}$ and $(V)_{\tilde{c},D,\rho}$ with $(\sqrt{3}-1)/2 < \rho \leq 1/2$ are fulfilled. Then the modified wave operators*

$$\tilde{\Omega}_{a,1} = \operatorname*{s-lim}_{t\to\infty} \tilde{U}_c(t)^* U_{a,1}(t)(P^a \otimes \mathrm{Id})$$

exist for all $a \subset c$, and are asymptotically complete

$$L^2(X) = \sum_{a \subset c} \oplus \operatorname{Ran} \tilde{\Omega}_{a,1}.$$

Let $a \subset c$. Since $\mathbf{D}_{\tilde{H}_{aD}(t)}(p_a - \tilde{b}(t)) = \mathbf{D}_{H_{a,1}(t)}(p_a - \tilde{b}(t)) = 0$, we have the following propagation properties of $\tilde{U}_{aD}(t)$ and $U_{a,1}(t)$.

Lemma 5.3. *The following estimates hold for $\phi \in \mathscr{D}(p^2+x^2)$ as $t \to \infty$:*

$$\begin{aligned} &\big\| |p_a - \tilde{b}(t)| \tilde{U}_{aD}(t)\phi \big\|_{L^2(X)} = O(1), \\ &\big\| |p_a - \tilde{b}(t)| U_{a,1}(t)\phi \big\|_{L^2(X)} = O(1). \end{aligned}$$

Corollary 5.4. *The following estimates hold for $\phi \in \mathscr{D}(p^2+x^2)$ as $t \to \infty$:*

$$\begin{aligned} &\big\| F_{\varepsilon_1/2}(t^{-1}|p_a - \tilde{b}(t)| \geq \varepsilon_1/2) \tilde{U}_{aD}(t)\phi \big\|_{L^2(X)} = O(t^{-1}), \\ &\big\| F_{\varepsilon_1/2}(t^{-1}|p_a - \tilde{b}(t)| \geq \varepsilon_1/2) U_{a,1}(t)\phi \big\|_{L^2(X)} = O(t^{-1}). \end{aligned}$$

By these estimates, we have the following.

Proposition 5.5. *The strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \tilde{U}_{aD}(t)^* U_{a,1}(t)$$

exists and is unitary on $L^2(X)$.

Proof. We put $\eta_a(t) = F_{\varepsilon_1/2}(t^{-1}|p_a - \tilde{b}(t)| \leq \varepsilon_1)$. By Corollary 5.4, we have only to prove the existence of the limits

$$\lim_{t\to\infty} \tilde{U}_{aD}(t)^*\eta_a(t) U_{a,1}(t)\phi, \quad \lim_{t\to\infty} U_{a,1}(t)^*\eta_a(t)\tilde{U}_{aD}(t)\phi$$

for $\phi \in \mathscr{D}(p^2+x^2)$. Noting

$$\begin{aligned} &I_c(p_a t + \tilde{c}(t) - t\tilde{b}(t))\eta_a(t) = I_c(t, p_a t + \tilde{c}(t) - t\tilde{b}(t))\eta_a(t), \\ &\mathbf{D}_{H_a(t)}(\eta_a(t)) = -t^{-2}F'_{\varepsilon_1/2}(t^{-1}|p_a - \tilde{b}(t)| \leq \varepsilon_1)|p_a - \tilde{b}(t)|, \end{aligned}$$

we obtain the proposition by virtue of Lemma 5.3. □

Combining Corollary 5.2 and Proposition 5.5 with Theorem 2.1, Theorem 1.2 can be obtained immediately.

6 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Throughout this section, we assume that $c = a_{\min}$ and that $(V)_{c,L}$ and $(V)_{\tilde{c},D,\rho}$ with $1/\{2(j_0+1)\} < \rho < 1/(2j_0)$ for some $j_0 \in \boldsymbol{N}$ are fulfilled. The case where $\rho = 1/(2j_0)$ can be included in the $1/\{2(j_0+1)\} < \rho < 1/(2j_0)$ by making ρ slightly smaller than $1/(2j_0)$. Since the proof is quite similar to the one in Adachi-Tamura [AT2], we sketch it with minor modification.

We construct an approximate solution of the Hamilton-Jacobi equation

$$(\partial_t S)(t,\xi) = \frac{1}{2}\xi^2 - E(t)\cdot(\nabla_\xi S)(t,\xi) + I_c(t, (\nabla_\xi S)(t,\xi)) \tag{6.1}$$

associated with $\tilde{H}_c(t)$. Putting $K(t,\xi) = S(t,\xi + \tilde{b}(t))$, (6.1) is translated into

$$(\partial_t K)(t,\xi) = \frac{1}{2}(\xi + \tilde{b}(t))^2 + I_c(t, (\nabla_\xi K)(t,\xi)). \tag{6.2}$$

Thus we will construct an approximate solution of (6.2). $K_0(t,\xi)$ denotes the solution of

$$(\partial_t K_0)(t,\xi) = \frac{1}{2}(\xi + \tilde{b}(t))^2, \quad K_0(0,\xi) = 0.$$

As mentioned in §1, $K_0(t,\xi)$ is written by (1.9), and (1.10) holds. We further define $K_j(t,\xi)$, $1 \le j \le j_0$, for $t \ge T$ inductively as the solution of

$$(\partial_t K_j)(t,\xi) = \frac{1}{2}(\xi + \tilde{b}(t))^2 + I_c(t, (\nabla_\xi K_{j-1})(t,\xi)), \quad K_j(T,\xi) = K_{j-1}(T,\xi).$$

Noting $(\partial_t K_0)(t,\xi) = (\xi + \tilde{b}(t))^2/2$, we have

$$K_j(t,\xi) = K_0(t,\xi) + \int_T^t I_c(\tau, (\nabla_\xi K_{j-1})(\tau,\xi))\,d\tau, \quad t \ge T \tag{6.3}$$

for $1 \le j \le j_0$. We here note that

$$\sup_{\xi \in X} |\partial_\xi^\beta (K_j(t,\xi) - K_{j-1}(t,\xi))| = O(t^{1-2j\rho}) \tag{6.4}$$

holds for $1 \le j \le j_0$ by virtue of (5.1), which can be proved by the Faà di Bruno formula and induction in j.

Putting $S_j(t,\xi) = K_j(t,\xi - \tilde{b}(t))$, $S_{j_0}(t,\xi)$ satisfies

$$(\partial_t S_{j_0})(t,\xi) = \frac{1}{2}\xi^2 - E(t)\cdot(\nabla_\xi S_{j_0})(t,\xi) + I_c(t, (\nabla_\xi S_{j_0-1})(t,\xi)). \tag{6.5}$$

We will write $I_c(t, (\nabla_\xi S_j)(t,\xi))$ as $I_{c,j}(t,\xi)$ below. We define the Hamiltonian $\hat{H}_c(t)$ by

$$\hat{H}_c(t) = H_c(t) + I_{c,j_0-1}(t,p)$$

for $t \ge T$, whose definition is slightly different from the one in [AT2]. $\hat{U}_c(t)$ denotes the propagator generated by $\hat{H}_c(t)$ such that $\hat{U}_c(T) = \mathrm{Id}$.

Lemma 6.1. *The following estimates hold for $\phi \in \mathscr{D}(p^2 + x^2)$ as $t \to \infty$:*

$$\| |x - (\nabla_\xi S_{j_0-1})(t,p)| \tilde{U}_c(t)\phi \|_{L^2(X)} = O(t^{1-2j_0\rho}), \tag{6.6}$$
$$\| |x - (\nabla_\xi S_{j_0-1})(t,p)| \hat{U}_c(t)\phi \|_{L^2(X)} = O(t^{1-2j_0\rho}). \tag{6.7}$$

For the proof, see [A4]. Since $g(t,x,p) = O(t^{-(2\rho+1)})$ and $r(t,x,p) = O(t^{-(2\rho+1)})$,

$$\begin{aligned} &I_c(t,x) - I_{c,j_0-1}(t,p) \\ &= O(t^{-(2\rho+1)})(x - (\nabla_\xi S_{j_0-1})(t,p)) + O(t^{-(2\rho+1)}) + O(t^{-2(j_0+1)\rho}) \end{aligned}$$

holds by virtue of (6.4). By this and Lemma 6.1, the following proposition can be obtained immediately, because $-(2\rho+1) + (1 - 2j_0\rho) = -2(j_0+1)\rho < -1$ and $-(2\rho+1) < -1$ by assumption.

Proposition 6.2. *The strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \hat{U}_c(t)^* \tilde{U}_c(t)$$

exists and is unitary on $L^2(X)$.

We would like to replace $\hat{U}_c(t)$ by

$$\begin{aligned} \check{U}_c(t) &= U_c(t,0)e^{-i\int_0^t I_c((\nabla_\xi K_0)(\tau,p))\,d\tau}, \quad t \geq 0, \quad \text{if } j_0 = 1, \\ \check{U}_c(t) &= U_c(t,0)e^{-i\int_T^t I_c((\nabla_\xi K_{j_0-1})(\tau,p))\,d\tau}, \quad t \geq T, \quad \text{if } j_0 \geq 2. \end{aligned} \tag{6.8}$$

We note that $\check{U}_c(t)$ is the propagator generated by the time-dependent Hamiltonian

$$\check{H}_c(t) = H_c(t) + I_c((\nabla_\xi S_{j_0-1})(t,p)).$$

We here used $U_c(t,0)pU_c(t,0)^* = p - \tilde{b}(t)$. We need the following lemma and corollary.

Lemma 6.3. *The following estimates hold for* $\phi \in \mathscr{D}(p^2+x^2)$ *as* $t \to \infty$:

$$\left\| |p - \tilde{b}(t)| \hat{U}_c(t)\phi \right\|_{L^2(X)} = O(1), \tag{6.9}$$

$$\left\| |p - \tilde{b}(t)| \check{U}_c(t)\phi \right\|_{L^2(X)} = O(1), \tag{6.10}$$

$$\left\| |(\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)| \hat{U}_c(t)\phi \right\|_{L^2(X)} = O(t), \tag{6.11}$$

$$\left\| |(\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)| \check{U}_c(t)\phi \right\|_{L^2(X)} = O(t). \tag{6.12}$$

Corollary 6.4. *The following estimates hold for* $\phi \in \mathscr{D}(p^2+x^2)$ *as* $t \to \infty$:

$$\left\| F_{\varepsilon_1/2}(t^{-2}|(\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)| \geq \varepsilon_1/2)\hat{U}_c(t)\phi \right\|_{L^2(X)} = O(t^{-1}), \tag{6.13}$$

$$\left\| F_{\varepsilon_1/2}(t^{-2}|(\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)| \geq \varepsilon_1/2)\check{U}_c(t)\phi \right\|_{L^2(X)} = O(t^{-1}). \tag{6.14}$$

By these results, we have the following.

Proposition 6.5. *The strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \check{U}_c(t)^* \hat{U}_c(t)$$

exists and is unitary on $L^2(X)$.

Proof. We put $\eta(t) = F_{\varepsilon_1/2}(t^{-2}|(\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)| \leq \varepsilon_1)$. By Corollary 6.4, we have only to prove the existence of the limits

$$\lim_{t\to\infty} \check{U}_c(t)^*\eta(t)\hat{U}_c(t)\phi, \quad \lim_{t\to\infty} \hat{U}_c(t)^*\eta(t)\check{U}_c(t)\phi$$

for $\phi \in \mathscr{D}(p^2+x^2)$. Putting $a(t,\xi) = F'_{\varepsilon_1/2}(t^{-2}|(\nabla_\xi S_{j_0-1})(t,\xi) - \tilde{c}(t)| \leq \varepsilon_1)((\nabla_\xi S_{j_0-1})(t,\xi) - \tilde{c}(t))/|(\nabla_\xi S_{j_0-1})(t,\xi) - \tilde{c}(t)|$, $\mathbf{D}_{H_c(t)}(\eta(t))$ is calculated as

$$\begin{aligned} &\mathbf{D}_{H_c(t)}(\eta(t)) \\ &= a(t,p)\cdot\{-2t^{-3}((\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)) \\ &\qquad + t^{-2}((\partial_t\nabla_\xi S_{j_0-1})(t,p) + E(t)\cdot(\nabla_\xi^2 S_{j_0-1})(t,p) - \tilde{b}(t))\} \\ &= a(t,p)\cdot\{-2t^{-3}((\nabla_\xi S_{j_0-1})(t,p) - \tilde{c}(t)) + t^{-2}(p - \tilde{b}(t) + (\nabla_\xi I_{c,j_0-2})(t,p))\}, \end{aligned}$$

where we used (6.5). Therefore the proposition can be obtained by virtue of Lemma 6.3. □

Combining Propositions 6.2 and 6.5 with Theorem 2.1, Theorem 1.3 can be obtained immediately.

参考文献

[A1] Adachi, T.: Long-range scattering for three-body Stark Hamiltonians. J. Math. Phys. **35**, 5547–5571 (1994)

[A2] Adachi, T.: Propagation estimates for N-body Stark Hamiltonians. Ann. Inst. H. Poincaré Phys. Théor. **62**, 409–428 (1995)

[A3] Adachi, T.: Scattering theory for N-body quantum systems in a time-periodic electric field. Funkcial. Ekvac. **44**, 335–376 (2001)

[A4] Adachi, T.: Asymptotic completeness for N-body quantum systems with long-range interactions in a time-periodic electric field. Comm. Math. Phys. **275**, 443–477 (2007)

[AT1] Adachi, T., Tamura, H.: Asymptotic Completeness for Long-Range Many-Particle Systems with Stark Effect. J. Math. Sci., The Univ. of Tokyo **2**, 77–116 (1995)

[AT2] Adachi, T., Tamura, H.: Asymptotic Completeness for Long-Range Many-Particle Systems with Stark Effect, II. Comm. Math. Phys. **174**, 537–559 (1996)

[AH] Avron, J.E., Herbst, I.W.: Spectral and scattering theory of Schrödinger operators related to the Stark effect. Comm. Math. Phys. **52**, 239–254 (1977)

[B] Brezis, H.: Analyse fonctionnelle, Théorie et applications. Paris: Masson 1983

[CFKS] Cycon, H., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger Operators with Application to Quantum Mechanics and Global Geometry. Springer-Verlag 1987

[D] Dereziński, J.: Asymptotic completeness of long-range N-body quantum systems, Ann. of Math. **138**, 427–476 (1993)

[DG1] Dereziński, J., Gérard, C.: Scattering theory of classical and quantum mechanical N-particle systems. Springer-Verlag 1997

[DG2] Dereziński, J., Gérard, C.: Long-range scattering in the position representation. J. Math. Phys. **38**, 3925–3942 (1997)

[EV] Enss, V., Veselić, K.: Bound states and propagating states for time-dependent Hamiltonians, Ann. Inst. H. Poincaré Sect. A (N.S.) **39**, 159–191 (1983)

[G] Gérard, C.: Sharp propagation estimates for N-particle systems. Duke Math. J. **67**, 483–515 (1992)

[GŁ] Gérard, C., Łaba, I.: Multiparticle Quantum Scattering in Constant Magnetic Fields. AMS 2002

[Gr1] Graf, G.M.: Phase space analysis of the charge transfer model. Helv. Phys. Acta **64**, 107–138 (1990)

[Gr2] Graf, G.M.: Asymptotic completeness for N-body short-range quantum systems: a new proof. Comm. Math. Phys. **132**, 73–101 (1990)

[Gr3] Graf, G.M.: A remark on long-range Stark scattering. Helv. Phys. Acta **64**, 1167–1174 (1991)

[HeSj] Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et équation de Harper. In: Lecture Notes in Physics **345**: Springer-Verlag (1989), pp. 118–197

[HMS1] Herbst, I., Møller, J.S., Skibsted, E.: Spectral analysis of N-body Stark Hamiltonians. Comm. Math. Phys. **174**, 261-294 (1995)

[HMS2] Herbst, I., Møller, J.S., Skibsted, E.: Asymptotic completeness for N-body Stark Hamiltonians. Comm. Math. Phys. **174**, 509–535 (1996)

[Ho1] Howland, J.S.: Scattering theory for time-dependent Hamiltonians, Math. Ann. **207**, 315–335 (1974)

[Ho2] Howland, J.S.: Scattering theory for Hamiltonians periodic in time, Indiana Univ. Math. J. **28**, 471–494 (1979)

[Hu] Hunziker, W.: On the space-time behavior of Schrödinger wavefunctions, J. Math. Phys. **7**, 300–304 (1966)

[JO] Jensen, A., Ozawa, T.: Existence and non-existence results for wave operators for perturbations of the Laplacian. Rev. Math. Phys. **5**, 601–629 (1993)

[JY] Jensen, A., Yajima, K.: On the long-range scattering for Stark Hamiltonians. J. Reine Angew. Math. **420**, 179–193 (1991)

[KY] Kitada, H., Yajima, K.: Scattering theory for time-dependent long-range potentials. Duke Math. J. **49**, 341–376 (1982)

[Mø] Møller, J.S.: Two-body short-range systems in a time-periodic electric field, Duke Math. J. **105**, 135-166 (2000)

[MøS] Møller, J.S., Skibsted, E.: Spectral theory of time-periodic many-body systems. Adv. Math. **188**, 137-221 (2004)

[M] Mourre, E.: Absence of singular continuous spectrum for certain self-adjoint operators. Comm. Math. Phys. **78**, 391–408 (1981)

[N] Nakamura, S.: Asymptotic completeness for three-body Schrödinger equations with time-periodic potentials. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33**, 379–402 (1986)

[P] Perry, P.A.: Scattering Theory by the Enss Method. Math. Rep. Vol.1, Harwood Academic, 1983

[RaS] Radin, C., Simon, B.: Invariant Domains for the Time-Dependent Schrödinger Equation. J. Differential Equations **29**, 289–296 (1978)

[RS] Reed, M., Simon, B.: Methods of Modern Mathematical Physics I–IV, Academic Press

[SS] Sigal, I.M., Soffer, A.: The N-particle scattering problem: asymptotic completeness for short-range systems. Ann. of Math. **125**, 35–108 (1987)

[Sk] Skibsted, E.: Propagation estimates for N-body Schroedinger Operators. Comm. Math. Phys. **142**, 67–98 (1991)

[W] White, D.: Modified wave operators and Stark Hamiltonians, Duke Math. J. **68**, 83–100 (1992)

[Y] Yafaev, D.R.: Wave operators for the Schrödinger equation. Theoret. and Math. Phys. **45**, 992–998 (1980)

[Ya1] Yajima, K.: Schrödinger equations with potentials periodic in time. J. Math. Soc. Japan **29**, 729–743 (1977)

[Ya2] Yajima, K.: Existence of solutions for Schrödinger evolution equations. Comm. Math. Phys. **110**, 415–426 (1987)

[YK] Yajima, K., Kitada, H.: Bound states and scattering states for time periodic Hamiltonians. Ann. Inst. H. Poincaré Sect. A (N.S.) **39**, 145–157 (1983)

[Yo1] Yokoyama, K.: Mourre theory for time-periodic systems, Nagoya Math. J. **149**, 193–210 (1998)

[Yo2] Yokoyama, K.: Asymptotic completeness for Hamiltonians with time-dependent electric fields. Osaka J. Math. **36**, 63–85 (1999)

[Z] Zielinski, L.: A proof of asymptotic completeness for N-body Schrödinger operators. Comm. Partial Differential Equations **19**, 455–522 (1994)

[Zo] Zorbas, J.: Scattering theory for Stark Hamiltonians involving long-range potentials. J. Math. Phys. **19**, 577–580 (1978)